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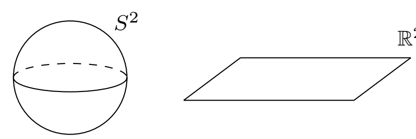
The Fundamental Group

Motivation

To motivate the subject let us first discuss what the main question of topology is. One can describe topology as the study of geometry on its lowest level, where it no longer matters what specifics of the form a geometric object has. As we settle this subject of discussion, the very first natural questions that comes up is how can we distinguish between two topological spaces. If the spaces are the same, as we know, all we have to do is to construct homeomorphism between two spaces (at least in most of the cases). However, harder is the question of proving that there exist no homeomorphisms between two spaces. In point-set topology, there are several ways with which one can show that two spaces cannot be topologically identical. Most of those ways are based upon the following quite simple idea: find a property that purely depends on the topology of the space, then check whether two spaces possess the property or not, and finally if one of them has the property, the other does not then they are topologically distinct. For instance, take the sphere (S^2) and the plane ($\mathbb{R}^2$).



The figure provides us with a visual sense that these two spaces must be different; however, to be precise we need a proof. To prove the distinction, we use the property of compactness*. S^2 is compact, since it is a bounded, closed subspace of $\mathbb{R}^3$, and $\mathbb{R}^2$ is not compact.

Here is another example; consider the plane ($\mathbb{R}^2$) and the line ($\mathbb{R}$). The standard trick here is to first assume that there exists a homeomorphism $\varphi : \mathbb{R} \to \mathbb{R}^2$. Then it can be trivially checked that $\overline{\varphi} : \mathbb{R}\backslash\{0\} \to \mathbb{R}^2\backslash\{\varphi(0)\}$ is also a homeomorphism. However, $\mathbb{R}\backslash\{0\}$ is not connected, and $\mathbb{R}^2\backslash\{\varphi(0)\}$ is connected regardless where $\varphi(0)$ ends up being. Finally, an example of a property which is *not* topological in nature is completeness of a metric space. For instance, take $S^2\backslash\{*\}$, where $* \in S^2$, and $\mathbb{R}^2$. We know that these two spaces are homeomorphic via the stereographic projection. Notice that $S^2\backslash\{*\}$ being a subspace of $\mathbb{R}^3$ inherits a metric from it, and under this metric the space is not complete due to the fact that it is not a closed subspace of $\mathbb{R}^3$. However, as we know $\mathbb{R}^2$ is complete under its standard metric.

Now let us go back to topological distinction of $\mathbb{R}^2$ and $\mathbb{R}$. Suppose that one tries to go one step beyond. Specifically, he asks the question of topological distinction of $\mathbb{R}^2$ and $\mathbb{R}^3$. Our first intuition, will tell us that these spaces must be distinct, and we will be right. However, if one tries to tackle this problem with the techniques from point-set topology, he will stumble across much hardship. In fact, this is the point where the subject of algebraic topology comes into play. The techniques in this subject can solve this problem and many more. The main trick of the subject is to assign algebraic objects (groups, rings, modules, etc.) to topological spaces, which again depend purely on the topology of the space. Furthermore, the language of algebraic objects and structural maps between them (homomorphisms, linear maps, etc.) fits quite "naturally" with the language of topological spaces and structural maps there (continuous maps, quotient maps, etc.). We will specify what we mean by this in a greater detail later. Why do we use algebraic objects? The reason for this is that we know a lot of information about those and that they are easier to deal with.

Before going into the subject let me bring a preview of another idea which is commonly used in algebraic topology. Topologists, and mathematicians in general, tend to look at how other objects map in and out of it instead of looking at the inner structure of the object . For instance, instead of saying that a group is simple† one may say that a homomorphism coming out of this group is either trivial or an imbedding. In fact, knowing the nature of continuous maps, in certain cases, one can already establish difference between them. A vivid and simple example of this statement, can be another proof of distinction of $\mathbb{R}$ and $\mathbb{R}^2$. Consider the circle (S^1) and the maps from it into these two spaces. There exist an injective continuous map from S^1 to $\mathbb{R}^2$. Now let us consider an arbitrary continuous map $f : S^1 \to \mathbb{R}$. The circle is compact, and therefore,

*As a reminder, the space is compact, if any open cover of that space has a finite subcover.
†Namely, it does not have a normal proper subgroup other than the trivial one.

im$f \subset \mathbb{R}$ is compact as well. A compact subspace of $\mathbb{R}$ is both closed and bounded. Therefore, im$f$ has a maximal and minimal points, which we reach at points α and β on S^1. If $\alpha = \beta$, then the function will be a constant one, clearly not injective. Otherwise, if $\alpha \neq \beta$, then there are two arcs connecting α and β, A and B. A and B are connected and they contain α and β; therefore, $[f(\beta), f(\alpha)] \subset f(A)$ and $[f(\beta), f(\alpha)] \subset f(B)$. However, this implies that f cannot be injective.

Definitions

In the previous example we have shown distinction of two topological spaces by considering maps from the circle into those spaces. A specific subset of these maps modulo some relation will be our main subject of discussion today. It will become the fundamental group as we add the algebraic structure on top of it. Instead of giving these vague qualitative definitions let us try to define this group precisely. However, our starting point will be the definition of the notion of a *path*.

Definition. A path in a topological space X is a continuous map[‡] $f : I \to X$, *where I is the compact interval of unit length,* $[0, 1]$. *A path homotopy between two paths f_0 and f_1 is a continuous function $F : I^2 \to X$, such that $F|_{i\times I} = f_i$ and $F|_{I\times i}$ is constant, for $i \in \{0, 1\}$. If for paths f and g there exists a path homotopy then we write that $f \simeq g$.*

The notion of a path is intuitively clear. Let us instead focus on the notion of path homotopy. One can informally conjure a dynamical picture of path homotopy. One can think of the first coordinate in I^2 as being the time. Then f_t, defined to be $f|_{t\times I}$, are paths in X, such that the endpoints of the paths do not move. Then the initial path and the final path are path homotopic. As a consequence, one can notice that if $f \simeq g$, then the endpoints of f and g are the same.

Proposition 1. The relation $\simeq$ is an equivalence relation, called path homotopy equivalence.

Proof. There are three properties to check here. The first one is the reflexivity, i.e. $f \simeq f$. Indeed, the path homotopy $F(t, s) = f(s)$ establishes this relation. If the condition $f \simeq g$ is satisfied, it follows that there exist a path homotopy F from f to g. Then $G(t, s) = F(1 - t, s)$ is path homotopy from g to f, and therefore, $g \simeq f$. Thus, the symmetry is satisfied. The last thing to show is the transitivity. Suppose F is a path homotopy from f to g, and G from g to h. Then define

$$H(t, s) = \begin{cases} F(2t, s) & \text{if } t \leq \frac{1}{2} \\ G(2t - 1, s) & \text{if } t \geq \frac{1}{2} \end{cases}.$$

One check that actually H is a path homotopy from f to h. □

This means we can define equivalence classes on the set of paths according to the relation $\simeq$. We will call such an equivalence class a *homotopy class*. We will write the set of homotopy classes on X by $\mathfrak{H}(X)$. At this point we actually can determine $\mathfrak{H}(\mathbb{R}^2)$ and we will do it! According to the comment made previously if $f \simeq g$ then the endpoints of f and g must be the same. In fact, for $\mathbb{R}^2$ this condition is also sufficient. Indeed, the homotopy will be the following $f_t(x) = (1 - t)f(x) + tg(x)$, where the multiplication and sums are performed upon vectors in $\mathbb{R}^2$. Therefore, we conclude that an element $\mathfrak{H}(\mathbb{R}^2)$ is one-to-one correspondence with pairs of points in $\mathbb{R}^2$.

Corollary 2. $\mathfrak{H}(\mathbb{R}^2) = \mathbb{R}^2 \times \mathbb{R}^2$.

Notice that we have not yet defined any algebraic structure in our subject of discussion. Let us begin by defining a multiplication on paths. Take two paths in space X, f and g, such that $f(0) = g(1)$. We define multiplication $\cdot$ between them as follows:

$$(f \cdot g)(s) = \begin{cases} g(2s) & \text{if } s \leq \frac{1}{2} \\ f(2s - 1) & \text{if } s \geq \frac{1}{2} \end{cases}.$$

However, notice that not all of the paths can be multiplied to one another. In addition, the multiplication on the paths will not be associative. This is not a circumstance that we desire. In order to achieve a much manageable situation we can do the following. First, we fix the issue of associativity. We realize that the multiplication is compatible with the homotopy equivalence relation. By this statement we mean that if $f \simeq f'$ and $g \simeq g'$, then $f \cdot g \simeq f' \cdot g'$. This fact can be shown quite easily.

[‡]From this point on all the maps between topological spaces are assumed to be continuous unless stated otherwise.

From this statement, we conclude that we can extend this multiplication on the homotopy classes. Here is how we are going to do it. As we know that for each homotopy class F there exists a notion of a starting point, F_0, and an endpoint, F_1. Suppose now that G is another class, such that $F_0 = G_1$. Now we can multiply these two classes in the following way. Pick representatives $f \in F$ and $g \in G$. Then $f(0) = F_0 = G_1 = g(1)$ and we can multiply f with g. Finally, we define $F \cdot G$ to be the class $f \cdot g$. This multiplication is well defined, since the multiplications on paths is compatible with $\simeq$.

Notice that not all the elements in $\mathfrak{H}(X)$ can be multiplied to one another. To correct this problem we choose a point, $x \in X$. We restrict our attention to the following subset of $\mathfrak{H}(X)$, $\{F \in \mathfrak{H}(X) | F_0 = F_1 = x\}$, which we will write as $\pi(X, x)$. Notice that any two elements in $\pi(X, x)$[§] can be multiplied and it is also multiplicatively closed. Even more, the following theorem holds.

Theorem 3. *$\pi(X, x)$ is a group, which is commonly referred to as the fundamental group of X.*

Proof. The multiplication is associative. We need to show that $(f \cdot g) \cdot h \simeq f \cdot (g \cdot h)$. The path homotopy is the following

$$F(t,s) = \begin{cases} f\left(\frac{4s}{1+t}\right) & \textit{if} \;\; s \le \frac{1+t}{4} \\ g\left(4s - t - 1\right) & \textit{if} \;\; \frac{1+t}{4} \le s \le \frac{2+t}{4} \\ h\left(\frac{4s-2-t}{2-t}\right) & \textit{if} \;\; s \ge \frac{2+t}{4} \end{cases}.$$

The reader may check that this is indeed a path homotopy between the paths mentioned above.

The role of the identity will be taken by the homotopy class of the constant map at x, γ. We have to show that $\gamma \cdot f \simeq f$ and $f \cdot \gamma \simeq f$, for any path that starts and ends at x. The following path homotopy:

$$G(t,s) = \begin{cases} f\left(\frac{2s}{1+t}\right) & \textit{if} \;\; s \le \frac{1+t}{2} \\ x & \textit{if} \;\; s \ge \frac{1+t}{4} \end{cases}$$

establishes the relation $f \cdot \gamma \simeq f$. The other relation follows through in an analogous way.

The last fact we need to show in order to prove the theorem is that the inverses exist. Suppose that f is a representative of a path homotopy class. Let $\overline{f}$ be a path defined by the equation $\overline{f}(s) = f(1-s)$. We show that $f \cdot \overline{f} \simeq \gamma$. The homotopy of paths that we are looking for is the following:

$$H(t,s) = \begin{cases} f\left(2ts\right) & \textit{if} \;\; s \le \frac{1}{2} \\ f\left(2t(1-s)\right) & \textit{if} \;\; s \ge \frac{1}{2} \end{cases}.$$

Finally, $\overline{f} \cdot f = \overline{f} \cdot \overline{\overline{f}} \simeq \gamma$. □

This group is extremely useful in determining differences between topological spaces. In addition to this, it is well compatible with "categorical" structure of topological spaces. However, before bringing forth some examples let me first settle a couple of questions.

First one is not a question, but rather an observation. In the proof for the theorem, notice that it is not necessary for the paths to start and end at the same point. One can state that multiplication on $\mathfrak{H}(X)$ is associative. What about the identity? Identities exist, and yes, there will be more than one. In fact, there is an identity for each point, $x \in X$. That identity will be the path class of the constant path at x, γ_x. If f and g are two paths, such that they can be multiplied to γ_x from right and left respectively, then $f \cdot \gamma_x \simeq f$ and $\gamma_x \cdot g \simeq g$. As a matter of fact, the elements in $\mathfrak{H}(X)$ can also be inverted. For each path f in X that starts at x and ends at y, we can define $\overline{f}$ just like in the proof of the theorem. Notice that $\overline{f}$ is left and right "multipliable" with f. Furthermore, $f \cdot \overline{f} \simeq \gamma_y$ and $\overline{f} \cdot f \simeq \gamma_x$. One can show that the inverse path class is unique using pure algebra. Thus, almost all the group axioms (or their analogs) are satisfied in $\mathfrak{H}(X)$ except that not all elements in it can be multiplied to one another. A structure such as this bears the name of *groupoid*. $\mathfrak{H}(X)$ is commonly referred to as the *fundamental groupoid*. In certain situations the fundamental groupoid is much easier to manipulate and it also contains the fundamental group as a part of it.

The second question is actually a question, however, linguistic in nature. Notice the use of article "the" before the term "fundamental", in the statement of the theorem, even though $\pi(X, x)$ is explicitly dependent on the choice of x. In general, it is not necessarily true that the group is independent of the choice of x. It

[§] In literature, this algebraic object is commonly written as $\pi_1(X, x)$. This notation comes from homotopy theory, where it is known as the first homotopy group.

will be correct to use the term "a fundamental group based at point x". The following theorem though will help us justify the original terminology.

Theorem 4. *If X is path-connected, then the fundamental groups based at any two points will be isomorphic to one another.*

Proof. Pick arbitrary two points x and y in X. Using path-connectedness of X, we can state that there exists a path h starting at x and ending at y. Let H be the path class of h. Now we can define the following map $\beta : \pi(X,x) \to \pi(X,y)$, such that $\beta(F) = HFH^{-1}$[¶]. This map is a homomorphism, $\beta(FG) = HFGH^{-1} = HFH^{-1}HGH^{-1} = \beta(F)\beta(G)$. The homomorphism is also bijective with the inverse, $\alpha(F) = H^{-1}FH$. Therefore, β is an isomorphism. □

If the space X is path-connected then we will sometimes, if it is not essential, omit the basepoint and simply write $\pi(X)$ for the fundamental group.

Now as we have established, at the very least definition-wise, what the fundamental group is, we will proceed to the examples and some facts about it.

Basic Facts, Examples, and Applications

We can already determine $\pi(\mathbb{R}^2)$. The corollary in the previous section tells informed us that $\mathfrak{H}(\mathbb{R}^2) = \mathbb{R}^2 \times \mathbb{R}^2$, where the first coordinate corresponds to the starting point of the class and the second coordinate to the endpoint. Hence, there exist one class $\mathfrak{H}(\mathbb{R}^2)$ that starts and ends at a given point and we conclude that $\pi(\mathbb{R}^2)$ is trivial. We can generalize this statement. If $\mathfrak{H}(X) = X \times X$ in the sense above, then $\pi(X)$ is trivial. Notice that omitting the basepoint is legitimate in this situation, since there exists a path between any two points in X. In fact, the converse statement is also true.

Proposition 5. *Suppose X is a path-connected space. Then, $\pi(X) = 0$ if and only if $\mathfrak{H}(X) = X \times X$.*

Proof. The part that need the proof is the direct statement. Suppose $\pi(X) = 0$, and suppose that F and G are two path classes with the same boundary points. We can write F as $FG^{-1}G$. However, FG^{-1} starts and ends at the same point; therefore, it can be considered as an element of the fundamental group at that point, and hence, FG^{-1} is an identity. Thus, $FG^{-1}G$ is also equal to G. In conclusion, $F = G$. □

The spaces that satisfy the conditions of proposition above are called *simply-connected*.

Notice that the same corollary in the previous chapter can be easily generalized for any $\mathbb{R}^n$, and using the proposition above we conclude with a corollary.

Corollary 6. $\pi(\mathbb{R}^n) = 0$ *for all* $n \in \mathbb{N}$. □

We can use this fact to prove the following rather non-trivial proposition.

Corollary 7. $\pi(S^n) = 0$ *if* $n \geq 2$.

Proof. Suppose that f is a path in S^n. Our goal is to show that this path is homotopic to the constant map. Suppose that f is not surjective. Then f can be considered as a path in $\mathbb{R}^n \cong S^n \backslash \{*\}$, where $* \in S^n$ is a point not in the image of f. In $\mathbb{R}^n$ we can collapse f to the constant map according to the previous corollary. Clearly, then f can be collapsed to a single point S^n.

In general, f is not required to be non-surjective and suppose it is not. We claim that there will always be g in S^n, such that $f \simeq g$ and g is not surjective; therefore, will be homotopic to the constant map. Choose $* \in S^n$, and a neighborhood of B of it which is homeomorphic to O^n, open ball in $\mathbb{R}^n$ centered at the origin. $f^{-1}(B)$ is an open subspace of I; hence, it is a union (not necessarily finite) of disjoint open intervals $\{I_\alpha\}_{\alpha \in \mathcal{J}}$. Clearly, $f^{-1}(*)$ is covered by this collection. $f^{-1}(*)$ is a closed subspace of a compact space; therefore, it is compact. Thus, there exists a finite subcollection, $\{I_n\}_{n=1}^k \subset \{I_\alpha\}_{\alpha \in \mathcal{J}}$, covering $f^{-1}(*)$. Suppose that $\varphi : \overline{B} \to D^n$ is a homeomorphism, such that carries $\varphi(*)$ is the center. Suppose $\sigma : J \to D^n$ is continuous map, where J is a compact interval within I, such non of the endpoints maps to the center. Let $\widetilde{\sigma} : J \to D^n$ be continuous map, such that does not go through the center and the endpoint match to that of σ. It will exist, since $n \geq 2$. Clearly, $\sigma \simeq \widetilde{\sigma}$ in a way similar to homotopy of paths.

Let $K = \bigcup I_n$. Let g be a path in S^n, such that $g|_{I \setminus K} = f|_{I \setminus K}$ and $g|_{\overline{I_n}} = \iota \circ \varphi^{-1} \circ \left(\varphi \circ f|_{\overline{I_n}}\right)^{\sim}$, where $\iota : \overline{B} \to S^n$ is the inclusion map, and $f|_{\overline{I_n}}$ is considered as a function from $\overline{I_n}$ to $\overline{B}$. It is easy to check that the function is well-defined. It is continuous by pasting lemma [JRM, 18.4]. In fact, $g \simeq f$ and $*$ is not in the image of g. □

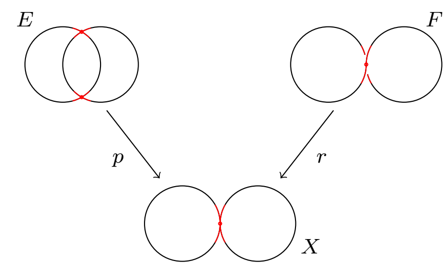
[¶]We will omit the dots if it does not result in confusion.

Proving that the fundamental group of any given space is trivial is much easier to prove the group is not trivial, since we have to show the lack of homotopy between paths. There are very elegant techniques of computing those types of groups using Seifert van Kampen theorem and the covering spaces. We provide a proof for the next theorem that uses the idea of covering spaces.

Theorem 8. $\pi(S^1) = \mathbb{Z}$.

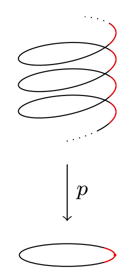
Proof. This is the covering space proof of the theorem. A covering of a space, X, is a space along with a projection map, such that the projection map defines a local homeomorphism between the covering space and the original space. More precisely, a covering of X is a continuous surjection $p : E \to X$, such that for $x \in X$ there exists an open neighborhood B of it, such that $p^{-1}(U) = \bigcup U_\alpha$, the elements of $\{U_\alpha\}$ are pairwise disjoint, and $p|_{U_\alpha}$ is a homeomorphism between U_α and U for each α. We will call U_α to be a *slice* of U.

The figure shows an example of a covering (E, p) defined in the following way. The large arcs, that connect the intersection points of two circles in E, wrap around the left circle in X, and the small ones wrap around the right circle. The reader should convince himself that this is indeed a covering. However, (F, r) is not a covering. Let us denote (as usual) the wedge-point of X with $*$. In the figure you can see a typical neighborhood of $*$, say B, painted in red. B has the shape of a cross. On the other hand, as you can see from the figure $r^{-1}(B)$ is a disjoint union of two open arcs each of them containing a single point of $r^{-1}(*)$. Open arcs are not homeomorphic to the cross and from this one can easily derive that r will not satisfy the conditions of the covering projection.



Covering spaces of a space are closely linked to its fundamental group. Even more, using the fundamental group one can determine up to covering isomorphism all possible coverings of the space. The statement can go the other way around: one can determine the fundamental group using the covering spaces. That is what we are going to do for S^1. The reader does not need to know the underlying theory behind covering spaces in order to be able to understand the proof. On the other hand, the proof provides a good insight into the theory of the covering spaces. We have delayed the actual proof long enough, so without further ado let us dive into the actual proof.

Consider the regular circular helix in $\mathbb{R}^3$, described by the parametrization $(\cos(2\pi s), \sin(2\pi s), s)$. This helix is homeomorphic to $\mathbb{R}$; therefore, we will denote this helix by $\mathbb{R}$. It projects down to the circle on the horizontal plane, which we will naturally write as S^1. Denote this projection by $p : \mathbb{R} \to S^1$. In fact $(\mathbb{R}, p)$ is a covering. As you can see from the picture if we take an arc around a point in S^1 small enough it lifts via p to a countable union of arcs in $\mathbb{R}$. First thing we are going to show in this proof is the following lemma, which is commonly known as *homotopy lifting property* of covering spaces. We actually can prove it for general covering spaces.



Lemma. Suppose that $p : E \to X$ is a covering. Given a map $F : M \times I \to X$ and a map $f : M \to E$, such that $pf = F_0$, there exists a unique map $\widetilde{F} : M \times I \to E$, such that $\widetilde{F}_0 = f$ and $p\widetilde{F} = F$.

Now notice that to show the existence and uniqueness of $\widetilde{F}$ as a function it is sufficient to show it if M is a point set. For this situation we temporarily reformulate the statement of the lemma: given a path F in X and a point σ in E, such that $p(\sigma) = F_0$, there exists a unique path $\widetilde{F}$ in E, such that $\widetilde{F}_0 = \sigma$ and $p\widetilde{F} = F$. For each $x \in X$ choose a neighborhood U_x, such that the pre-image satisfies the covering property. The collection $\{U_x\}$ is an open cover of X; therefore, $\{F^{-1}(U_x)\}$ is an open cover of I. Using Lebesgue number lemma [JRM, 27.5], we can say that there is a finite collection of open intervals $\{I_n\}_{n=0}^k$, such that $\bigcup I_n = I$ and $F(I_n) \subset U_n \in \{U_x\}$. We can enumerate the set $\{I_n\}$, so that $0 \in I_0$ and $I_n \cap I_{n+1} \neq \emptyset$. Now $\sigma \in p^{-1}(F_0) \subset p^{-1}(F(I_0)) \subset p^{-1}(U_0)$. Thus, we define V_0 to be the slice of U_0 that contains σ. $F(I_n \cap I_{n+1}) \subset F(I_n) \cap F(I_{n+1}) \subset U_n \cap U_{n+1}$; thus, $U_n \cap U_{n+1} \neq \emptyset$. Thus, there exists a unique slice V_{n+1} of U_{n+1}, such that $V_{n+1} \cap V_n \neq \emptyset$. Now we define a set of functions $G_m : \mathfrak{I}_m \to E$, where $\mathfrak{I}_m = \bigcup_{n=1}^m I_n$, such that $G_m|_{I_n} = i_n \left(p|_{V_n}\right)^{-1} F|_{I_n}$, where $i_n : V_n \to E$ is the inclusion map. Notice $G_m(0) = \sigma$ and $pG_m = F|_{\mathfrak{I}_m}$; in fact, we claim that this function is the only one with these two properties. We prove this claim by induction. Suppose we are given a continuous function $G : I_0 \to E$, such that $G(0) = \sigma$

and $pG = F|_{I_0}$. As we know $F(I_0) \subset U_0$; therefore, $G(I_0) \subset p^{-1}(U_0)$. However, $G(I_0)$ is connected; therefore, $G(I_0) \subset V_0$, since $\sigma \in G(I_0)$. Thus, we can write the following equation $(p|_{V_0})G = F|_{I_0}$, of course, bearing in mind the restrictions of the ranges. Since, $p|_{V_0}$ is invertible, we conclude that $G = i_0(p|_{V_0})^{-1}F|_{I_0}$. Suppose the claim is true for $m-1$, and suppose that $G : \mathfrak{I}_m \to E$, such that $G(0) = \sigma$ and $pG = F|_{\mathfrak{I}_m}$. Notice that the inductive assumption implies that $G|_{\mathfrak{I}_{m-1}} = G_{m-1}$. Notice $G(I_m) \subset p^{-1}(U_m)$, and since $G(I_m)$ is connected, we conclude that $G(I_m)$ lies within a single slice of U_m. Remember $J = I_{m-1} \cap I_m \neq \emptyset$. Combine $G(J) = G_{m-1}(J) \subset V_{m-1}$ with $G(J) \subset G(I_{m-1}) \cap G(I_m)$ to obtain that $V_{m-1} \cap G(I_m) \neq \emptyset$. Thus, $G(I_m) \subset V_m$. We rewrite $(p|_{V_m})G|_{I_m} = F|_{V_m}$ and get that $G|_{I_m} = i_m(p|_{V_m})^{-1}F|_{I_m}$. This clearly implies that $G|_{I_m} = G_m$. This proves the claim. To prove the existence and uniqueness of $\widetilde{F}$ just set $m = k$. Before going any further let me comment on a certain property of $\widetilde{F}$. If $J \subset I$ is an interval, such that $F(J) \subset U$, where U is open and satisfies the property of the covering projection and V is a slice of it, such that $\widetilde{F}(J) \subset V$, then $\widetilde{F}|_J = i(p|_V)^{-1}F|_J$, where $i : V \to E$ is the inclusion map.

Now let us go back to the general case. The discussion above basically showed that there exist a unique function $\widetilde{F} : M \times I \to E$, such that $\widetilde{F}_0 = f$, $p\widetilde{F} = F$ and $\widetilde{F}|_{m \times I}$ is continuous for each $m \in M$. Hence, in order to prove the lemma we are only left to show that $\widetilde{F}$ is continuous itself. Pick an arbitrary $\mu \in M$. Consider the open cover $\{F^{-1}(U_x)\}$. One can show, using the compactness of I, that there exists a neighborhood L of μ and a finite collection of intervals $\{I_n\}_{n=0}^k$, such that $F(L \times I_n) \subset U_n \in \{U_x\}$. In addition, we order those intervals, so that $0 \in I_0$ and $I_n \cap I_{n+1} \neq \emptyset$. f is continuous; hence, there exists an open $N \subset L$, such that $f(N)$ lies within a single slice of U_0, say V_0. Thus, $\widetilde{F}(N \times I_0) \subset V_0$. Therefore, we can conclude that $\widetilde{F}|_{N \times I_0} = i_0(p|_{V_0})^{-1}F|_{N \times I_0}$ and $\widetilde{F}|_{N \times I_0}$ is continuous. Using induction like previously, we can show that $\widetilde{F}|_{N \times I_n}$ is continuous as well. Since μ was arbitrary, we conclude that $\widetilde{F}$ is continuous.

Let $*$ denote the point $(1, 0)$ in S^1. From lemma it follows that that it is possible to lift uniquely a path ω in S^1 based at $*$ to a path $\widetilde{\omega}$ in $\mathbb{R}$ based at $(1, 0, 0)$. We define the function $\Phi : \pi(X, *) \to \mathbb{Z}$ that maps the class of the loop f to the z-coordinate of $\widetilde{f}(1)$. From the lemma it also follows that if $f \simeq g$, then $\widetilde{f} \simeq \widetilde{g}$. This shows that Φ is well-defined, since $\widetilde{f} \simeq \widetilde{g}$ implies that $\widetilde{f}(1) = \widetilde{g}(1)$. In fact, Φ is surjective. Indeed, for $n \in \mathbb{Z}$ consider the path $v(s) = (\cos(2\pi ns), \sin(2\pi ns), ns)$: $\Phi(pv)$ is the z-coordinate $\widetilde{pv}(1) = v(1) = n$. Furthermore, Φ is injective. If $f \not\simeq g$, then $\widetilde{f} \not\simeq \widetilde{g}$, and therefore $\widetilde{f}(1) \neq \widetilde{g}(1)$ and so are are they z-coordinates. Thus, the elements of $\pi(X, *)$ are the classes of the paths $\omega_n(s) = (\cos(2\pi ns), \sin(2\pi ns))$. With this we have determined what $\pi(X, *)$ is as a set. The final step of the proof will be showing that the multiplication in $\mathbb{Z}$ is the addition. Consider the following homotopy

$$H(s,t) = \begin{cases} \omega\left(\frac{2(n+m)ns}{(m-n)t+2n}\right) & \textit{if} \;\; s \le \frac{t}{2} + \frac{n(1-t)}{n+m} \\ \omega\left(\frac{2(n+m)m(s-1)}{(n-m)t+2m}\right) & \textit{otherwise} \end{cases},$$

where $\omega(s) = (\cos(2\pi s), \sin(2\pi s))$. Notice $H_0 = \omega_{m+n}$ and $H_1 = \omega_m \cdot \omega_n$; therefore, we conclude that $\omega_{m+n} \simeq \omega_m \cdot \omega_n$. Thus, the multiplication in $\pi(X, *)$ corresponds to regular addition in $\mathbb{Z}$. We are done. Huaaa! $\square$

Surprisingly, this theorem becomes computationally very powerful if we add several other theorems to it. However, just the way it is right now, one can conclude several very elegant and well-known results. One of those results is the Fundamental Theorem of Algebra.

Theorem 9. *Every nonconstant polynomial with coefficient in $\mathbb{C}$ has a root in $\mathbb{C}$.*

Proof. Suppose a polynomial $p(z) = z^n + a_{n-1}z^{n-1} + ... + a_0$ has no solutions. Define the following paths on S^1:

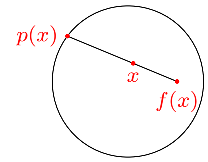
$$f_r(\varphi) = \frac{p(re^{2\pi i\varphi})/p(r)}{|p(re^{2\pi i\varphi})/p(r)|}.$$

f_r under this definition is a loop in $S^1 \subset \mathbb{C}$ that is based at 1. Since, f_r is continuous with respect to r and f_0 is the constant loop at 1. Thus, the homotopy path class of f_r is the identity class in $\pi_1(S^1)$ for all $r \in \mathbb{R}$. Suppose now that $r > \max\{|a_{n-1}| + ... + |a_0|, 1\}$, then if $|z| = r$

$$|z^n| = |z||z^{n-1}| > |z^{n-1}|(|a_{n-1}| + ... + |a_0|) \ge |a_{n-1}z^{n-1} + ... + a_0|.$$

In conclusion, $p_t(z) = z^n + t(a_{n-1}z^{n-1} + ... + a_0)$ has no solution if $|z| = r$ and $0 \le t \le 1$. Now if we replace p with p_t in the definition of f_r and vary t from 1 to 0, we obtain a homotopy between f_r and ω_n, where $\omega_n(s) = e^{2i\pi ns}$. This is only possible if $n = 0$, or the polynomial is a constant polynomial. $\square$

Yet another famous consequence is the Brouwer fixed point theorem.



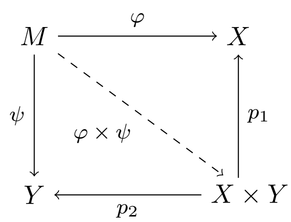
Theorem 10. Every continuous map $f : D^2 \to D^2$ has a fixed point.

Proof. Suppose that on the contrary that there exists a map without a fixed point. Then for each point on $x \in D^2$, we take a ray starting at $f(x)$ and passing through x. We take the intersection of this ray with $\partial D^2 \cong S^1$ and call it $p(x)$. We choose $p(x)$ so that x is in between $p(x)$ and $f(x)$. $p : D^2 \to S^1$ is a continuous function, such that $p|_{S^1}$ is the identity map. We can say the latter in the following way: if $i : S^1 \to D^2$ is the inclusion map of the boundary circle into the disk, then $pi = \mathbb{1}_{S^1}$. Choose a path r representing a non-identity class of $\pi_1(S^1)$. Then ir is a path in D^2; therefore, it is homotopic to the constant map. However, $pir = r$ is a path in S^1 homotopic to the constant path. This is impossible, since we have chosen r to be not of the identity class. □

The fundamental group behaves quite nicely with products of spaces.

Proposition 11. If X and Y are path-connected, then $\pi(X \times Y)$ is isomorphic to $\pi(X) \times \pi(Y)$.

Proof. The diagram below summarizes the fundamental property of the product of spaces.



Given any space M and any two maps φ and ψ as in the diagram, there exists a unique $\varphi \times \psi$ as in the diagram, such that $p_1 \circ (\varphi \times \psi) = \varphi$ and $p_2 \circ (\varphi \times \psi) = \psi$, where p_1 and p_2 denote the canonical projection maps. We construct the following maps

$$\mathcal{P}(X \times Y) \underset{\beta}{\overset{\alpha}{\rightleftarrows}} \mathcal{P}(X) \times \mathcal{P}(Y)$$

Here $\mathcal{P}(S)$ denotes the set of paths in S. We define $\alpha(r) = (p_1 r, p_2 r)$ and $\beta(f, g) = f \times g$. These two are inverse maps to one another: $\alpha\beta(f, g) = (p_1(f \times g), p_2(f \times g)) = (f, g)$ and $\beta\alpha(r) = p_1 r \times p_2 r = r$ by the uniqueness property. One can check that α and β are compatible with multiplication and the homotopy equivalence relation, which means that they can be extended to a pair of inverse homomorphism between $\pi(X \times Y)$ and $\pi(X) \times \pi(Y)$, thus, proving that they are isomorphic. □

Functorial Properties of the Fundamental Group

So forth we freely stated that if two topological spaces are homeomorphic to one another then their fundamental groups (based at corresponding points) must be the same. We are going to show that this fact formally and in the process we will introduce a more general notion of *induced map*. Suppose we are given a map $\varphi : X \to Y$, such that it maps x to y. Suppose we are given a homotopy class $A \in \pi(X, x)$. Pick a representative of it, $\alpha \in A$. Let us remember that α is a continuous map from I to A, such that it start and ends at x. Similarly, $\varphi \circ \alpha$ is continuous map from I to Y, which starts and ends at y. The homotopy class of $\varphi \circ \alpha$ therefore will be an element of $\pi(Y, y)$, which we will write as $\varphi_*(A)$. However, what we have just done may not be legitimate in some sense, since it may depend on the choice of the representative. Luckily, it does not, since if $\alpha \simeq \gamma$, then $\varphi \circ \alpha \simeq \varphi \circ \gamma$. We conclude that φ_* is a function from $\pi(X, x)$ to $\pi(Y, y)$. Now the question is whether this map is a homomorphism or not. Suppose β is a loop based at x. It is an easy check that $\varphi \circ (\alpha \cdot \beta) = (\varphi \circ \alpha) \cdot (\varphi \circ \beta)$. Therefore, φ_* is indeed a homomorphism.

The functorial properties mention in the title of the section are the following:

- If we have the following sequence of continuous maps $(X, x) \xrightarrow{\varphi} (Y, y) \xrightarrow{\psi} (Z, z)$, then $(\psi\varphi)_* = \psi_* \varphi_*$,
- $(\mathbb{1}_X)_* = \mathbb{1}_{\pi(X,x)}$.

The term "functorial" comes from category theory, in which the fundamental group is characterized as a functor from the category of topological space to the category of groups. From these properties one can easily deduce that if φ is a homeomorphism then φ_* is a group isomorphism. Indeed, φ and φ^{-1} are continuous; thus, $\varphi_*(\varphi^{-1})_* = (\varphi\varphi^{-1})_* = \mathbb{1}_* = \mathbb{1}$ and $(\varphi^{-1})_*\varphi_* = (\varphi^{-1}\varphi)_* = \mathbb{1}$. With this we formally have established that two homeomorphic spaces have the same fundamental groups.

Let me pause for a second to address a couple of caveats. If φ is injective or surjective, then it does *not* imply the same property for φ_*. Here are examples for each of the cases. Consider the imbedding $\sigma : S^1 \to D^2$ by mapping to the boundary. σ_* maps $\mathbb{Z}$ to the trivial group; therefore, there is no way it can be injective. For the surjectivity consider the following map $\mu : I \to S^1$, such that $\mu(s) = e^{2i\pi s}$. μ is surjective; however, μ_* maps the trivial group to $\mathbb{Z}$. No way it can be surjective.

Despite these caveats, there is a condition under which the induced homomorphism will be injective.

Proposition 12. Suppose $i : S \to X$ is injective, and there exist a continuous map $r : X \to S$, called the retract of i, such that $ri = \mathbb{1}_S$, then i_* is injective.

Proof. Simply notice that $r_* i_* = \mathbb{1}_{\pi(S)}$ is injective; therefore, i_* is also injective. □

This is a useful criterion to determine that a space does not retract to a subspace. However, it is also true that the retracts of a space can provide valuable information about the fundamental group of the whole space. At this point we found that the fundamental groups of retracts imbed into the fundamental group of the whole space. We can do better though with a special type of a retract.

Definition. A subspace $R \subset X$ is deformation retract if there exists a continuous map $F : X \times I \to X$, such that $F_0 = \mathbb{1}_X$, F is constant on $R \times I$ and $F_1(X) \subset R$.

In fact, if we have a deformation retract, its inclusion map is an isomorphism. In particular, this implies that $R^n \backslash \{0\}$ has the same fundamental group as S^{n-1}[‖], since the latter is a deformation retract of the former. Indeed, $F(\mathbf{x}, t) = \mathbf{x}\left(\frac{t}{|\mathbf{x}|} + (1-t)\right)$ is the required map. Thus, we obtain get the following proposition.

Proposition 13.

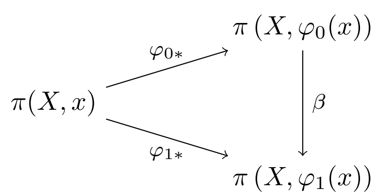
$$\pi(\mathbb{R}^n \backslash \{0\}) = \begin{cases} \mathbb{Z} & \text{if } n = 2 \\ 0 & \text{otherwise} \end{cases}.$$

From this proposition we immediately conclude with a corollary.

Corollary 14. $\mathbb{R}^2$ is not homeomorphic to $\mathbb{R}^n$ unless $n = 2$.

Now, to support the statements above we prove a general fact.

Proposition 15. Suppose $\varphi : X \times I \longrightarrow Y$. Then following diagram commutes, i.e. $\beta\varphi_{0*} = \varphi_{1*}$, where β is the isomorphism from theorem 4 for the path $\varphi_t(x)$.



Proof. Consider the following homotopy:

$$F(s,t) = \begin{cases} \varphi_{3s}(x) & \text{if } s \leq \frac{t}{3} \\ \varphi_t f\left(\frac{3s-t}{3-2t}\right) & \text{if } \frac{t}{3} \leq s \leq 1 - \frac{t}{3} \\ \varphi_{3(1-s)}(x) & \text{if } s \geq 1 - \frac{t}{3} \end{cases}.$$

Notice that $F_0 = \varphi_0 f$. If we call the path $\varphi_s(x)$ by ω, then we can write $F_1 \simeq \overline{\omega}\cdot\varphi_1 f\cdot\omega$. Thus, $\varphi_0 f \simeq \overline{\omega}\cdot\varphi_1 f\cdot\omega$. It is a good exercise for the reader to check that with this the theorem is proved. □

[‖]It is true even if $n = 1$. Even though S^0 has two path components both of them have the same fundamental group, the trivial one.

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